

INVARIANT NONRECURRENT FATOU COMPONENTS OF AUTOMORPHISMS OF \mathbb{C}^2

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ABSTRACT. Let Ω be an invariant nonrecurrent Fatou component associated with the automorphism $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. Assume that all of the limit maps of $\{F^n|_\Omega\}$ are constant. We prove the following theorem. If there is more than one such limit map then there are uncountably many. The images of these limit maps form a closed set in the boundary of Ω containing no isolated points. Additionally there cannot be more than one limit map unless the derivative of F along a specific subset of the curve of fixed points of F has eigenvalues 1 and $e^{i2\pi\theta}$, with θ non-Diophantine.

We also examine the case where the limit maps are not all constant. The image of a nonconstant limit map is an immersed variety in the boundary of Ω . We show that any two such immersed varieties intersect either trivially or in a set that is open in their intrinsic topologies.

We present some examples of maps with invariant nonrecurrent Fatou components.

1. INTRODUCTION

The Fatou components for rational self maps of $\overline{\mathbb{C}}$ are entirely classified: see for example [CG93]. Fornæss and Sibony [FS95] have examined recurrent Fatou components of holomorphic self maps of \mathbb{P}^2 of degree at least 2. Ueda [Ued94] has also made contributions in this direction. Fornæss and Sibony have also [FS98] studied recurrent Fatou components for generic maximal rank k holomorphic self maps of \mathbb{C}^k . Bedford and Smillie [BS91a] and [BS91b] have investigated Fatou components of Hénon maps. In this paper we consider invariant nonrecurrent Fatou components, Ω , for an automorphism, F , of \mathbb{C}^2 .

We consider first the case where all the limit maps are rank 0. Here we show that generically there cannot be more than one such limit map. Next we examine the case where there are rank 1 limit maps. We show that the images of such maps generically do not intersect. We show next that a large class of polynomial automorphisms of \mathbb{C}^2 do not have invariant nonrecurrent Fatou components. Finally we provide some examples of maps with invariant nonrecurrent Fatou components.

In Section 2 we examine the case where all limit maps are rank 0. We say that such a Fatou component satisfies Property 0. We let J be the set of fixed points of F , Σ be the images of the limit maps of $\{F^n\}_{n=0}^\infty$. We construct an F invariant curve which lies entirely in Ω . If certain eigenvalues of F' along Σ satisfy the Diophantine condition (3), defined on page 6, we construct continuously varying families of invariant manifolds transversal to J .

Using the invariant curve and the invariant manifolds which we have constructed, we prove the following theorem.

THEOREM 1.1. *Let Ω be an invariant, nonrecurrent Fatou component satisfying Property 0. If Σ contains more than one point, then the following hold:*

- (1) Σ is uncountable and has no isolated points.
- (2) There is a one dimensional subvariety $V \subset \{(z, w) \in \mathbb{C}^2; F(z, w) = (z, w)\}$ such that $\Sigma \subset V \cap \partial\Omega$.
- (3) There exists an $\alpha \in \mathbb{C}$, with $\alpha = e^{i2\pi\theta}$, with θ non Diophantine, such that the eigenvalues of $F'(z, w)$ are $\{1, \alpha\}$ for all $(z, w) \in \Sigma$.

In Section 3 we consider the case where the limit maps may be either rank 0 or rank 1. In Section 3.3 we find a natural way to extend a rank 1 limit map, h , to its image, allowing us to examine the action of iterates of F on this image. We conclude that the family $\{F^n\}$ is normal on $h(\Omega)$, if we consider F^n as a map from $h(\Omega)$ to \mathbb{C}^2 . Using the fact that the images of Ω under rank 1 limit maps are immersed varieties, in Section 3.4 we prove the following theorem.

THEOREM 1.2. *Let $h(\Omega)$ and $g(\Omega)$ be two distinct rank 1 limit varieties. Then $h(\Omega) \cap g(\Omega)$ is either empty or an open set, when considered as a subset of $h(\Omega)$ or $g(\Omega)$.*

In Section 4 we examine the question of whether polynomial automorphisms can have periodic nonrecurrent Fatou components, and prove the following theorem.

THEOREM 1.3. *If F is a polynomial automorphism of \mathbb{C}^2 then it cannot have an invariant nonrecurrent Fatou component on which it has more than one rank 0 limit map.*

If F is a polynomial automorphism of \mathbb{C}^2 with an invariant nonrecurrent Fatou component on which it has one rank 0 limit, then it is a Hénon map.

If F is a polynomial automorphism of \mathbb{C}^2 with an invariant nonrecurrent Fatou component on which it has a rank 1 limit, then it is a nonhyperbolic Hénon map.

In Section 5 we present some examples of such maps. We present an example of an automorphism which has an invariant nonrecurrent Fatou component with exactly one rank 0 limit map. We give two examples of automorphisms with rank 1 limits. The first has precisely one rank 1 limit, with image the w axis. The second rank 1 example has multiple rank 1 limit maps, all with image the w axis.

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2. THE RANK 0 CASE

Let $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be an automorphism. Let the Fatou set, \mathcal{F} , denote the set of points where the forward iterates $\{F^n; n \geq 0\}$ are locally normal: every subsequence $\{F^{n_j}\}$ has a further subsequence that converges uniformly on compact subsets of \mathcal{F} . Let Ω denote a connected component of \mathcal{F} , i.e. a Fatou component. A Fatou component Ω is said to be

invariant if $F(\Omega) = \Omega$. We say that Ω is recurrent if there is a point, $z_0 \in \Omega$, and a sequence of integers $\{n_j\}$, such that $F^{n_j}(z_0) \rightarrow z \in \Omega$.

If Ω is a Fatou component, let Σ denote the set of maps, $h : \Omega \rightarrow \mathbb{C}^2$, which are obtained as normal limits of subsequences of $\{F^n|_\Omega\}$. If Ω is invariant but not recurrent, then every $h \in \Sigma$ maps Ω to $\partial\Omega$. In this section we consider the case where each $h \in \Sigma$ has rank zero, and is thus constant. We say in this case that Ω satisfies Property 0. We identify h with the point $h(\Omega) \in \partial\Omega$.

The remainder of Section 2 consists of a proof of Theorem 1.1.

LEMMA 2.1. *Let F , Ω and Σ be as in the statement of Theorem 1.1. Then we have the following.*

- (1) Σ contains uncountably many elements.
- (2) Σ is a closed set containing no isolated points.
- (3) Every point in Σ is a neutral fixed point.

Proof. The Lemma is proved by constructing an F invariant curve lying in Ω , which connects the points $\{F^n(p)\}$, for some $p \in \Omega$. Given the images of any two limit maps, the curve travels back and forth between these images, and clusters at the boundary of Ω . These cluster points are the uncountably many limit points mentioned in the Lemma. Using the local normal forms described by Ueda [Ued86] we show that each of these points is a neutral fixed point.

2.1. An Invariant Curve. We construct an F invariant curve γ which we use to show that if we have two limit maps in Σ then in fact we have uncountably many. Let $p \in \Omega$. Let $\gamma_0 : [0, 1] \rightarrow \mathbb{C}^2$ be any smooth curve from p to $F(p)$ which is entirely contained in Ω . Let $\gamma_n : [n, n+1] \rightarrow \mathbb{C}^2$ be $F^n(\gamma_0(t-n))$. Finally, let $\gamma : [0, \infty) \rightarrow \mathbb{C}^2$ be given by

$$\gamma(t) = \gamma_n(t) \text{ when } t \in [n, n+1].$$

Clearly γ is F invariant.

Let $q = \lim_{j \rightarrow \infty} (F|_\Omega)^{m_j}$ and $q' = \lim_{k \rightarrow \infty} F^{n_k}$ be two distinct points in Σ , and let $B^2(q, \epsilon)$ and $B^2(q', \epsilon)$ be two nonintersecting balls around q and q' . Notice that γ is connected. Also note that for large enough j we have $F^{m_j}(\gamma_0) \subset B^2(q, \epsilon)$ and $F^{n_k}(\gamma_0) \subset B^2(q', \epsilon)$. By the connectivity of γ there must be points of γ in $\partial B^2(q, \epsilon)$. We make this more precise.

Given any $t_0 \in [0, 1]$ there are an $m'_0 \in \{m_j\}$ and an $n'_0 \in \{n_k\}$ such that

- (1) $m'_0 < n'_0$,
- (2) $F^{m'_0}(\gamma_0(t_0)) = \gamma_{m'_0}(t_0 + m'_0) \in B^2(q, \epsilon)$, and
- (3) $F^{n'_0}(\gamma_0(t_0)) = \gamma_{n'_0}(t_0 + n'_0) \in B^2(q', \epsilon)$.

Then there is a t_1 such that

- (1) $m'_0 < t_1 < n'_0$, and
- (2) $\gamma(t_1) \in \partial B^2(q, \epsilon)$.

Repeating the above procedure, but choosing m'_1, n'_1 both bigger than n'_0 , we produce $m'_1 < t_2 < n'_1$ such that $\gamma(t_2) \in \partial B^2(q, \epsilon)$. We repeat again, producing the sequences $\{t_i\}_{i=0}^\infty, \{m'_i\}_{i=0}^\infty, \{n'_i\}_{i=0}^\infty$, and $\{\gamma(t_i)\}_{i=0}^\infty$. We note that for each t_i we have

$$\gamma(t_i) = \gamma_{l_i}(\zeta_i + l_i) = F^{l_i}(\gamma_0(\zeta_i)),$$

for some integer l_i and some $\zeta_i \in [0, 1]$. Passing to subsequences we can assume that

- (1) $\zeta_i \rightarrow \zeta \in [0, 1]$,
- (2) $\gamma(t_i) \rightarrow \eta_\epsilon \in \partial B^2(q, \epsilon)$, and
- (3) $(F|_\Omega)^{l_i}$ converges.

Then

$$\begin{aligned} \lim_{i \rightarrow \infty} F^{l_i}(\gamma_0(\zeta)) &= \lim_{i \rightarrow \infty} F^{l_i}(\gamma_0(\zeta_i)) \\ &= \lim_{i \rightarrow \infty} \gamma(t_i) \\ &= \eta_\epsilon. \end{aligned}$$

We know that $\eta_\epsilon \in \partial\Omega$: if it were not, Ω would be recurrent.

Given two limit maps q and q' we have produced a third, η_ϵ . We notice, however, that for each ϵ suitably small we find a different η_ϵ . We have thus shown that given two limit maps there are in fact uncountably many.

2.2. The Structure of F on Σ . By the construction of the previous section, given a limit map q , there are other limit maps arbitrarily close to q . In other words, Σ contains no isolated points.

We claim that Σ is a closed set. To prove this assume there is $q \in \overline{\Sigma} \setminus \Sigma$. Then, given $\epsilon > 0$, there is a point $q_\epsilon \in \Sigma$ such that

$$|q - q_\epsilon| < \epsilon/2.$$

Since $q_\epsilon \in \Sigma$, we can find n_ϵ so that for a fixed $p \in \Omega$ we have

$$|F^{n_\epsilon}(p) - q_\epsilon| < \epsilon/2.$$

But then

$$|F^{n_\epsilon}(p) - q| < \epsilon.$$

Passing to a convergent subsequence of $\{(F|_\Omega)^{n_\epsilon}\}$, say $\{(F|_\Omega)^{n_j}\}$, we see that $(F|_\Omega)^{n_j} \rightarrow q$.

Any point $q = \lim_{j \rightarrow \infty} (F|_\Omega)^{n_j} \in \Sigma$ is fixed:

$$\begin{aligned} F(q) &= F \lim_{j \rightarrow \infty} F^{n_j}(p) \\ &= \lim_{j \rightarrow \infty} F^{n_j}(F(p)) \\ &= q. \end{aligned}$$

We denote the set of fixed points of F by J . J is an analytic variety in \mathbb{C}^2 , so it is either an open set, or consists of a union of isolated points and one complex dimensional curves. The F we are examining cannot be the identity, so J cannot be open. We have also seen that Σ contains no isolated points. Thus we see that any point in Σ lies on a one dimensional

complex curve in J . According to Ueda [Ued86] Section 3 we can thus find local coordinates around any point $q \in \Sigma$ in which F takes the form

$$(1) \quad F(x, y) = (x + g(x, y)h(x, y), y + g(x, y)k(x, y)),$$

where $g(x, y)$ is a defining equation of J . Singular points of J are discrete, so we can find $q \in \Sigma$ at which J is not singular. Then, again according to Ueda [Ued86], we may choose local coordinates in which J is the x axis:

$$F(x, y) = (x + yh(x, y), y(1 + k(x, y))).$$

We note that along the x axis we have

$$DF(x, 0) = \begin{pmatrix} 1 & h(x, 0) \\ 0 & 1 + k(x, 0) \end{pmatrix}.$$

The eigenvalues of $DF(x, 0)$ are 1 and $1 + k(x, 0)$. There are three possibilities:

- (1) q is semi repulsive: $|1 + k(x, 0)| > 1$,
- (2) q is semi attractive: $|1 + k(x, 0)| < 1$, or
- (3) q is neutral: $|1 + k(x, 0)| = 1$.

To show that the former two cases are not possible, we note that Nishimura [Nis83] has shown that in the semi repulsive (resp. semi attractive) case F can be written, in suitable coordinates, as

$$(2) \quad F(x, y) = (x, b(x)y),$$

with $|b(x)| > 1$ (resp. < 1).

To show that q is not semi repulsive, assume that $|b(x)| > 1$ in some suitably small neighbourhood, V , of $q = (0, 0)$. Assume as well, by shrinking V if needed, that $|b|$ is bounded above on V and that V is a polydisk of polyradius ϵ . Fixing a point $(x, y) \in V \cap \Omega$, with $y \neq 0$, we know that

$$(x_{n_j}, y_{n_j}) = F^{n_j}(x, y) \rightarrow q,$$

for some subsequence $\{n_j\}$ of integers. So for all j suitably large we have that

$$|y_{n_j}| < \frac{\epsilon}{2 \max_{(x, y) \in \bar{V}} |b(x)|^2}.$$

Carefully choosing m_j , since $|b(x)| > 1$ uniformly on V and $y_{n_j} \neq 0$, we can arrange that

$$(x_{n_j+m_j}, y_{n_j+m_j}) = (x_{n_j}, (b(x_{n_j}))^{m_j} y_{n_j}) \in V,$$

and that

$$\frac{\epsilon}{|b(x_{n_j})|^2} \leq |y_{n_j+m_j}| \leq \frac{\epsilon}{|b(x_{n_j})|}.$$

This gives us a new sequence, $\{n_j + m_j\}$, where $F^{n_j+m_j}(x, y)$ lie in

$$\left\{ (x, y) \in V \mid \frac{\epsilon}{|b(x)|^2} \leq |y| \leq \frac{\epsilon}{|b(x)|} \right\}.$$

This set is a compact, so $\{F^{n_j+m_j}(x, y)\}$ has a limit point, q , in \bar{V} which is not in the x axis. If $q \in \Omega$ then Ω is a recurrent Fatou component, which we have assumed it is not. If

q is in the boundary of Ω , then since Ω satisfies Property 0 it is the image of a limit map of $\{(F|_{\Omega})^n\}_{n=1}^{\infty}$. By above results q is a fixed point of F . But, if V is small enough, the only fixed points in \bar{V} are on the x axis. We see that our fixed points cannot be semi repulsive.

Assume by way of contradiction that q is a semi attractive fixed point. It is clear by the form of Equation (2) that in a small neighbourhood of q we have convergence of the iterates of F to a map whose image contains an open set in the x axis. But we assumed that all limits of F were constant maps, so this situation is not possible. Another way of looking at this is that the normal form also shows that we have normality in a neighbourhood of our fixed point. But our fixed point is supposed to be in the boundary of the Fatou component.

Note that we have not accounted for points in Σ which are singular points of J . However, both by continuity and by Equation (1) we see that these points are also neutral.

This completes the proof of the lemma. \square

Remark 2.2. *The following remark holds unless the eigenvalues of F' are both constant along the curve of fixed points.*

We note that at smooth points of J we have that $1 + k(x, 0)$ is holomorphic: denote the eigenvalue $1 + k(x, 0)$ by $\lambda(z)$. Then $\lambda(z)$ is a root of the polynomial $P(\lambda, z) = \det(F'(z) - \lambda I)$. This polynomial has holomorphic coefficients, and the root 1. Thus P can be written as $(\lambda - 1)(\lambda - \lambda(z))$. We thus see that $\lambda(z)$ is the constant term of P , and thus holomorphic.

Now we recall that Ueda tells us that the eigenvalue in the direction of J is constant 1, and that the eigenvalue in the transverse direction to J is $1 + k(x, 0)$. On Σ we know that $|1 + k(x, 0)| = 1$, and thus $1 + k(x, 0)$ varies real analytically, away from singularities of J .

We notice that the above considerations show that Σ is contained in a locally finite union of local real analytic curves, with discrete singularities; precisely the curves where one eigenvalue of DF is exactly 1 and the second eigenvalue of DF has modulus 1.

We consider one-parameter families of (local) holomorphic diffeomorphisms of \mathbb{C}^2 and study the parametric dependence of local invariant manifolds. We study a case where the maps are not necessarily hyperbolic at the fixed point. This will be the key tool in the proof of Theorem 1.1.

We denote coordinates on \mathbb{C}^2 by $z = (z_1, z_2)$. We shall need the following Diophantine condition

$$(3) \quad |\lambda^k - 1| > ck^{-N}, \quad k = 1, 2, \dots,$$

where $\lambda = e^{2\pi i\theta}$.

THEOREM 2.3. *Let $F = F_r : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ be a family of local holomorphic diffeomorphisms with the following properties:*

- (1) $F(z) = (\lambda z_1 + O(|z|^2), z_2 + O(|z|^2))$,
- (2) F depends holomorphically on λ .

We restrict our attention to $\lambda = re^{2\pi i\theta_0}$, $1 - \delta < r < 1 + \delta$ for δ small to be chosen later, and $0 < \theta_0 < 1$ is a fixed irrational satisfying the Diophantine condition (3).

Then:

- (1) For any r in the above range there exists a local invariant manifold $\psi = \psi_r : \Delta(0, \rho) \rightarrow \mathbb{C}^2$ $F \circ \psi_r(w) = \psi_r(\lambda w)$, $\psi_r(0) = (0, 0)$ $\psi'_r(0) = (1, 0)$, for a fixed ρ independent of r .
- (2) There exist uniform bounds $|\psi_r|_{L_\infty(D(0, \rho))} < C$ independent of r .
- (3) The family ψ_r is C^1 -smooth in r in the compact-open topology of $\mathcal{O}(\Delta, \mathbb{C}^2)$.

Proof. The proof of the Theorem is a parameterized version of Siegel's linearization Theorem, and uses ideas from Pöschel's paper [Pös86].

Let M be a uniform bound on the modulus of F in a neighbourhood of the origin.

We use multi-index notation: $l = (l_1, l_2) \in \mathbb{N}^2$, $|l| = l_1 + l_2$, $z^l = z_1^{l_1} z_2^{l_2}$. Let

$$F(z) = \sum_{|l| \geq 1} \vec{f}_l z^l = \Lambda z + \sum_{|l| \geq 2} \vec{f}_l z^l,$$

where

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \vec{f}_l = (f_l^1, f_l^2) \in \mathbb{C}^2.$$

The $\vec{f}_l = \vec{f}_l(\lambda)$ depend on λ ; we suppress the λ .

We shall also use the following definitions:

$$\epsilon_n^1 = \lambda^n - \lambda, \quad \epsilon_n^2 = \lambda^n - 1 \quad \text{and} \quad \epsilon_n = \min(\epsilon_n^1, \epsilon_n^2) \quad \text{for } n \geq 2,$$

$$E_n = \lambda^n Id - \Lambda = \begin{pmatrix} \epsilon_n^1 & 0 \\ 0 & \epsilon_n^2 \end{pmatrix}.$$

For vectors in \mathbb{R}^2 define the usual lexicographic ordering

$$(z'_1, z'_2) \preceq (z''_1, z''_2) \text{ if and only if } z'_1 \leq z''_1 \text{ and } z'_2 \leq z''_2.$$

Extend this to formal power series in $\mathbb{R}^2[[w]]$ by

$$\sum \vec{a}_i w^i \preceq \sum \vec{b}_i w^i \text{ if and only if } \vec{a}_i \preceq \vec{b}_i \text{ for all } i.$$

For vectors in \mathbb{C}^2 denote as usual $\|z\| := \max(|z_1|, |z_2|)$ and by $\{\cdot\}$ the “norm” $\{(z_1, z_2)\} := (\|z\|, \|z\|)$. We view w as a formal object and extend the norms to $\mathbb{C}^2[[w]]$ by $\{\sum \vec{a}_i w^i\} := \sum \{\vec{a}_i\} w^i$. We also let the usual norm on \mathbb{C} act on $\mathbb{C}[[w]]$ by $|\sum \vec{a}_i w^i| := \sum |\vec{a}_i| w^i$. With these conventions observe that $\{A(w) + B(w)\} \preceq \{A(w)\} + \{B(w)\}$ and $|A(w)^l| \preceq \{A(w)\}^l$ for any formal power series $A(w)$, $B(w)$ with coefficients in \mathbb{C}^2 . For a diagonal matrix

$$A = \begin{pmatrix} a^1 & 0 \\ 0 & a^2 \end{pmatrix}$$

define

$$\{A\} = \begin{pmatrix} \|(a^1, a^2)\| & 0 \\ 0 & \|(a^1, a^2)\| \end{pmatrix}.$$

Observe that $\{A^{-1}\}^{-1}\{\vec{v}\} \preceq \{A\vec{v}\}$, for any vector \vec{v} .

We are looking for a map

$$(4) \quad \psi(w) = \sum_{k=1}^{\infty} \overrightarrow{\psi}_k w^k = Jw + \sum_{k=2}^{\infty} \overrightarrow{\psi}_k w^k,$$

where

$$J = \overrightarrow{\psi}_1 = (1, 0), \quad \overrightarrow{\psi}_k = (\psi_k^1, \psi_k^2) \in \mathbb{C}^2, \quad k \in \mathbb{N}, \quad w \in \mathbb{C},$$

satisfying the functional equation $F \circ \psi(w) = \psi(\lambda w)$. In power series this functional equation can be written

$$(5) \quad \sum_{n \geq 2} (\lambda^n Id - \Lambda) \overrightarrow{\psi}_n w^n = \sum_{|l| \geq 2} \overrightarrow{f}_l \left(\sum_{k \geq 1} \overrightarrow{\psi}_k w^k \right)^l.$$

Applying $\{\cdot\}$ to (5) gives

$$\begin{aligned} \sum_{n \geq 2} \{E_n \overrightarrow{\psi}_n\} w^n &\preceq \sum_{|l| \geq 2} \left\{ \overrightarrow{f}_l \left(\sum_{k \geq 1} \overrightarrow{\psi}_k w^k \right)^l \right\} \\ &\preceq \sum_{|l| \geq 2} \{ \overrightarrow{f}_l \} \left| \left(\sum_{k \geq 1} \overrightarrow{\psi}_k w^k \right)^l \right| \\ &\preceq \sum_{|l| \geq 2} \{ \overrightarrow{f}_l \} \left(\sum_{k \geq 1} \{ \overrightarrow{\psi}_k \} w^k \right)^l. \end{aligned}$$

We conclude that

$$(6) \quad \sum_{n \geq 2} \{E_n^{-1}\}^{-1} \{ \overrightarrow{\psi}_n \} w^n \preceq \sum_{|l| \geq 2} \{ \overrightarrow{f}_l \} \left(\sum_{k \geq 1} \{ \overrightarrow{\psi}_k \} w^k \right)^l.$$

Comparing powers of w in (5) it is possible to find $\overrightarrow{\psi}_n$ recursively:

$$\overrightarrow{\psi}_n = E_n^{-1} P_n(\overrightarrow{\psi}_1, \dots, \overrightarrow{\psi}_{n-1}, (\overrightarrow{f}_l)_{2 \leq |l| \leq n}) = E_n^{-1} (P_n^1, P_n^2), \quad n \geq 2.$$

The functions P_n^1, P_n^2 are polynomials in the coordinates of their vector arguments. The coefficients of these polynomials are positive. Note that the coefficients do not depend on λ and \overrightarrow{f}_l ; they are the same for all linearization problems in this context. For our purposes they are universal polynomials.

Equating coefficients in powers of w in (6) the same way as we did in (5) we can rewrite (6) as

$$(7) \quad \{ \overrightarrow{\psi}_n \} \preceq \{E_n^{-1}\} P_n(\{ \overrightarrow{\psi}_1 \}, \dots, \{ \overrightarrow{\psi}_{n-1} \}, (\{ \overrightarrow{f}_l \})_{2 \leq |l| \leq n}), \quad n \geq 2.$$

Define the polynomials Q_n in E_j^{-1} and \vec{f}_l recursively by

$$\begin{aligned} Q_1 &= (1, 0), \\ Q_n &= Q_n(E_2^{-1}, \dots, E_{n-1}^{-1}, (\vec{f}_l)_{2 \leq |l| \leq n}) \\ &= E_n^{-1} P_n(Q_1, \dots, Q_{n-1}, (\vec{f}_l)_{2 \leq |l| \leq n}), \quad n \geq 2. \end{aligned}$$

These are the polynomials which result in “unravelling” the recursive relations defining the $\vec{\psi}_n$. In other words, they exhibit the explicit dependence of $\vec{\psi}_n$ on the coefficients $(\vec{f}_l)_{|l| \geq 2}$ of F and the small divisors $\lambda^k - \lambda$, and $\lambda^k - 1$. That is to say $\vec{\psi}_n = Q_n$. Then from (7) and the definition of $\{\cdot\}$ it follows that

$$\begin{aligned} \|\vec{\psi}_n\| &\leq \|Q_n(\{E_2^{-1}\}, \dots, \{E_{n-1}^{-1}\}, (\{\vec{f}_l\})_{2 \leq |l| \leq n})\| \\ &\leq \|Q_n(\{E_2^{-1}\}, \dots, \{E_{n-1}^{-1}\}, ((M, M))_{2 \leq |l| \leq n})\|. \end{aligned}$$

We thus see that in order to exhibit a uniform exponential bound on $\vec{\psi}_n$ it is enough to exhibit such a bound with \vec{f}_l replaced by $\vec{M} = (M, M)$ and the small divisors $\epsilon_n^1, \epsilon_n^2$ replaced by ϵ_n in (6).

To do so we proceed as follows. Let $\vec{\sigma}_n = (\sigma_n, \sigma_n)$ be a sequence defined by setting $\vec{\sigma}_1 = (1, 1)$ and equating coefficients of powers of w in

$$(8) \quad \sum_{n \geq 2} \{(\lambda^n Id - \Lambda)^{-1}\}^{-1} \vec{\sigma}_n w^n = \sum_{|l| \geq 2} \vec{M} \left(\sum_{k \geq 1} \vec{\sigma}_k w^k \right)^l.$$

In other words

$$\begin{aligned} \vec{\sigma}_n &= \{E_n^{-1}\} P_n((\vec{\sigma}_k)_{2 \leq k \leq n-1}, (\vec{M})_{2 \leq |l| \leq n}) \\ &= Q_n(\{E_2^{-1}\}, \dots, \{E_{n-1}^{-1}\}, (\vec{M})_{2 \leq |l| \leq n}). \end{aligned}$$

Using the fact that P_n and Q_n are polynomials with positive coefficients it is clear that $\{\vec{\psi}_n\} \preceq \vec{\sigma}_n$. Now rewrite (8) in terms of σ_n :

$$\begin{aligned} \sum_{n \geq 2} \epsilon_n \sigma_n w^n &= M \sum_{|l| \geq 2} \left(\sum_{k \geq 1} \vec{\sigma}_k w^k \right)^l \\ &= M \sum_{\nu \geq 2} \sum_{l_1 + l_2 = \nu} \left(\sum_{k \geq 1} \sigma_k w^k \right)^\nu \\ &= M \sum_{\nu \geq 2} (\nu + 1) \left(\sum_{k \geq 1} \sigma_k w^k \right)^\nu \\ &= M \sum_{n \geq 2} w^n \sum_{k_1 + \dots + k_\nu = n, \nu \geq 2, k_i \geq 1} (\nu + 1) \sigma_{k_1} \dots \sigma_{k_\nu}. \end{aligned}$$

We conclude that

$$(9) \quad \sigma_n = \epsilon_n^{-1} M \sum_{\nu \geq 2, k_1 + \dots + k_\nu = n, k_i \geq 1} (\nu + 1) \sigma_{k_1} \cdots \sigma_{k_\nu}, \quad n \geq 2.$$

Up until now we have been working with $\lambda = r e^{i\theta_0}$ with an arbitrary fixed r . According to Lemma 2.5 below we can replace λ and the corresponding small divisors ϵ_n in (9) with the ones corresponding to $r = 1$, $\lambda_0 = e^{i\theta_0}$, by increasing M by a uniform factor. For simplicity we keep the same notation.

Following Pöschel-Brjuno-Siegel (see [Pös86] page 959) we split the problem into two, one involving no small divisors, and one involving only the small divisors. Let

$$\eta_1 = 1, \quad \eta_n = M \sum_{\nu \geq 2, k_1 + \dots + k_\nu = n, k_i \geq 1} (\nu + 1) \eta_{k_1} \cdots \eta_{k_\nu}$$

and

$$\delta_1 = 1, \quad \delta_k = \frac{1}{\epsilon_k} \max_{\nu \geq 2, k_1 + \dots + k_\nu = k, k_i \geq 1} \delta_{k_1} \cdots \delta_{k_\nu}, \quad k \geq 2, \quad n \geq 2.$$

It is easy to see by induction that $\sigma_n \leq \eta_n \delta_n$. We refer to Pöschel ([Pös86], pages 959-963) for the bound $\delta_n \leq C a^n$, $a = a(\theta_0)$, and restrict our attention to bounding η_n . These numbers satisfy

$$\sum_{n \geq 2} \eta_n w^n = M \sum_{\nu \geq 2} (\nu + 1) \left(\sum_{k \geq 1} \eta_k w^k \right)^\nu.$$

Setting $\eta = \eta(w) = \sum_{n \geq 1} \eta_n w^n$ the last equation becomes

$$\eta - w = M \sum_{\nu \geq 2} (\nu + 1) \eta^\nu = M \left(\frac{1}{(1 - \eta)^2} - 1 - 2\eta \right).$$

By the implicit function theorem this defines $\eta = \eta(w)$, $\eta(0) = 0$, as an analytic function in a disk $w \in D(0, 1/b)$. Here $\eta = \eta(w)$ depends on M only.

In particular, $\eta_n \leq C b^n$ for some $C > 0$ also dependent on M . Therefore $\|\psi_n\| \leq C(abM)^n$ independently of r . This, together with the observation that ψ_n are rational functions in λ and thus continuous in r for an irrational θ , shows that $\psi = \psi(\cdot; r)$ is a continuous family in $\mathcal{O}(D(0, \rho), \mathbb{C}^2)$, equipped with the compact-open topology. This $\rho = \frac{1}{abM}$ is the one mentioned in the statement of the theorem.

We now address the question of smoothness of this family. By the chain rule $\frac{d}{dr} = \frac{d\lambda}{dr} \frac{d}{d\lambda} = e^{2\pi i \theta} \frac{d}{d\lambda}$. We consider the formal derivative of the series ψ ,

$$\frac{d}{d\lambda} \psi = \sum_{n \geq 1} \frac{d\vec{\psi}_n}{d\lambda} w^n.$$

To prove convergence and continuity in r it is enough to demonstrate a uniform exponential bound on the coefficients $\frac{d}{d\lambda}\overrightarrow{\psi}_n$. To this end

$$\frac{d}{d\lambda}\psi_n^j = \frac{d}{d\lambda}P_n^j = \frac{d}{d\lambda}Q_n^j, \quad j = 1, 2.$$

As we observed previously, Q_n^j is a polynomial in $\overrightarrow{f}_l, (\epsilon_n^1)^{-1}, (\epsilon_n^2)^{-1}$ with positive coefficients:

$$(10) \quad Q_n^j = \sum A_{s,t,l,q} f_{l_1} \cdots f_{l_s} \epsilon_{q_1}^{-1} \cdots \epsilon_{q_t}^{-1}$$

where $(l = l_i)_{1 \leq i \leq s}$ and $q = (q_j)_{1 \leq j \leq t}$ are appropriate sequences of indexes¹ and $A_{s,t,l,q} \geq 0$, $|l_i| \leq n$, $|q_i| \leq n$. We refrain from giving more details on the explicit expansion of Q_n 's as this will require introducing a tree formalism to deal with their recursive definition and is not necessary for our purposes ([CF94]). What is important for us is that $s, t \leq n^2$: none of the monomials are of degree higher than $2n^2$. To prove this denote by d_n the maximum number of ϵ 's in a monomial of Q_n . Since $Q_1 = (1, 0)$ we have $d_n = 0$. We proceed by induction. Considering (5) we see that $d_n \leq 1 + \max(d_{k_1} + \cdots + d_{k_\nu})$ where the maximum extends over all $2 \leq \nu \leq n$, $k_i \geq 1$, $k_1 + \cdots + k_\nu = n$. Then inductively

$$d_n \leq 1 + \max(k_1^2 + \cdots + k_\nu^2) < 1 + n^2$$

and thus $d_n \leq n^2$. The last inequality follows since $k_i \geq 1$. Similarly we can prove that $t \leq n^2$.

Consider the $\frac{d}{d\lambda}$ derivative of (10). The coefficients $A_{s,t,q,l}$ are independent of λ and the product rule gives

$$\begin{aligned} \left| \frac{d}{d\lambda} Q_n^j \right| &= \left| \sum A_{s,t,q,l} f_{l_1} \cdots \frac{df_{l_i}}{d\lambda} \cdots f_{l_s} \epsilon_{q_1}^{-1} \cdots \epsilon_{q_t}^{-1} + \right. \\ &\quad \left. + \sum A_{s,t,q,l} f_{l_1} \cdots f_{l_s} \epsilon_{q_1}^{-1} \cdots \frac{d\epsilon_{q_k}^{-1}}{d\lambda} \cdots \epsilon_{q_t}^{-1} \right| \\ &\leq \sum A_{s,t,q,l} |f_{l_1}| \cdots \left| \frac{df_{l_i}}{d\lambda} \right| \cdots |f_{l_s}| |\epsilon_{q_1}^{-1}| \cdots |\epsilon_{q_t}^{-1}| + \\ &\quad + \max \left(\epsilon_{q_k} \frac{d\epsilon_{q_k}^{-1}}{d\lambda} \right) \sum A_{s,t,q,l} |f_{l_1}| \cdots |f_{l_s}| |\epsilon_{q_1}^{-1}| \cdots |\epsilon_{q_t}^{-1}| \\ &\leq C d_n |Q_n^j((M, M), \dots, \epsilon_n^{-1}, \dots)| \\ &\leq C d_n \sigma_n. \end{aligned}$$

The second to last inequality is easily deduced by an explicit computation of $\epsilon_{q_k} \frac{d\epsilon_{q_k}^{-1}}{d\lambda}$ and utilizing the Diophantine condition on λ .

¹We abuse notation here and consider f_{l_i} denoting a component of f_{l_i} rather than f_{l_i} itself. This is part of what we mean by "appropriate indexes". The other problem that we do not address explicitly is the index sets over which l_i vary. The same disclaimer applies to the index q_i of ϵ_{q_i}

Finally, since σ_n is bounded exponentially and uniformly so are the coefficients of the formally derived series $\frac{d}{d\lambda}\psi$. Thus convergence and continuity of $\frac{d\psi}{dr}$ in r are established. We conclude that the family of invariant manifolds is C^1 -smooth along the curve $\{\arg \lambda = \theta_0, 1 - \delta < r < 1 + \delta\}$. \square

Remark 2.4. *Using the same methods it is possible to show higher order smoothness of the family of invariant manifolds along the curve $\{\arg \lambda = \theta_0, 1 - \delta < r < 1 + \delta\}$. In fact, the family is C^∞ -smooth in $r = |\lambda|$. More general approach regions are possible too, provided a Diophantine-type condition remains true uniformly in r . However, the series ψ parametrizing the invariant manifolds are definitely not holomorphic in λ . Their coefficients are rational functions of λ containing all possible terms $\lambda^n - 1$ in denominators and thus explode at a dense set of points on the curve $\{|\lambda| = 1\}$.*

LEMMA 2.5. *There exist constants $c > 0$ and $N \geq 0$ independent of $1 - \delta < r < 1 + \delta$ such that for $\lambda = re^{2\pi i\theta_0}$*

$$|\lambda^k - 1| \geq \sqrt{2}/2 |e^{2\pi i k \theta_0} - 1| \geq ck^{-N} \quad \text{for all } k \geq 1.$$

Proof. It is enough to show that if $\lambda_0 = e^{2\pi i\theta_0} \in S^1$ satisfies (3) for c_0 and N_0 then the same estimates holds for all $\lambda = re^{2\pi i\theta_0}$ with r close to 1 for some $c > 0$, $N \geq 0$.

Let

$$I = \{k \in \mathbb{N} : |\arg(e^{2\pi i k \theta_0})| > \pi/4\}$$

where $-\pi < \arg(e^{2\pi i k \theta_0}) \leq \pi$.

For $k \in I$ we have

$$|r^k e^{2\pi i k \theta_0} - 1| > \sin(\pi/4) = \sqrt{2}/2.$$

For $k \in \mathbb{N} \setminus I$

$$\begin{aligned} |r^k e^{2\pi i k \theta_0} - 1| &\geq |e^{2\pi i k \theta_0} - 1| \cos(\arg(e^{2\pi i k \theta_0})) \geq \\ &|e^{2\pi i k \theta_0} - 1| \cos(\pi/4) = \sqrt{2}/2 |e^{2\pi i k \theta_0} - 1| \geq \sqrt{2}/2 ck^{-N}. \end{aligned}$$

\square

We now complete the proof of Theorem 1.1 The idea of the proof is as follows: We assume that the normal eigenvalue is not constant. Then, for certain points, x_1 and x_2 , lying in Σ we shall create paths, Γ_1 and Γ_2 . The Γ_i are to lie entirely in J , and will pass through x_i . At each point on Γ_i we will find an invariant manifold tranverse to J . This family of invariant manifolds will vary in a C^1 smooth fashion along Γ_i . The two families of manifolds thus built disconnect a small neighbourhood. Using the fact that the γ constructed in Lemma 2.1 does not intersect the two families of manifolds we will be able to create a recurrence in Ω . This contradiction will complete the proof.

We have shown that if there is more than one limit map then Σ lies in J , and the normal eigenvalue to J along Σ is of constant modulus 1. We focus our attention on smooth points of J in Σ , and use the local form described in Lemma 2.1, where J is the x axis. We assume, by way of contradiction, that the normal eigenvalue is not constant. For ease of notation

we shall call this eigenvalue $\lambda(x, 0)$ or $\lambda(x)$. We have noted before that at smooth points of J , $\lambda(x)$ is holomorphic.

We separate J into three sets J_s , J_u , and J_n on which F is respectively attracting, repelling or neutral in the normal direction:

$$\begin{aligned} J_s &= \{(x, 0) \in J \mid |\lambda(x, 0)| < 1\}, \\ J_u &= \{(x, 0) \in J \mid |\lambda(x, 0)| > 1\}, \text{ and} \\ J_n &= \{(x, 0) \in J \mid |\lambda(x, 0)| = 1\}. \end{aligned}$$

We now consider a generic point $(x_0, 0) \in J_n$: $(x_0, 0)$ is a smooth point of J where $\lambda(x_0) \neq 1$ and $\lambda'(x_0) \neq 0$. By the holomorphicity of λ and the implicit function theorem there is a small neighborhood of $(x_0, 0) \in U \subset \mathbb{C}^2$ where

$$J_n \cap U = \lambda^{-1}(S^1) \cap U = \{(x, 0) \in J \mid |\lambda(x)| = 1\} \cap U$$

is a smooth real one dimensional curve. By [Khi64] inequality (3) is satisfied on a full measure set $D' \subset S^1$. Thus $D = \lambda^{-1}(D') \subset J_n$ is dense in J_n , and we can also choose $x_0 \in D$, i.e. $\lambda(x_0) \in D'$, so that (3) is satisfied at $(x_0, 0)$ with some $c_0 > 0$, $N_0 \geq 0$. We consider the smooth real curve in J

$$\Gamma' = \{(x, 0) \in J \mid \arg \lambda(x) = \arg \lambda(x_0)\} \cap U.$$

Observe that $\lambda'(x_0) \neq 0$ implies Γ' is transversal to J_n at $(x_0, 0)$. If we choose Γ to be a small connected component of Γ' containing $(x_0, 0)$, then Γ does not intersect J_n , except at $(x_0, 0)$.

By Theorem 2.3 we obtain a family of local invariant manifolds, one passing through each point on Γ .

Choose an $x_0 \in \Sigma$ and a neighbourhood, $U(x_0)$, of x_0 in J satisfying the following:

- (1) J is smooth in $U(x_0)$,
- (2) $\lambda(x) \neq 1$ in $U(x_0)$,
- (3) $\lambda'(x) \neq 0$ in $U(x_0)$, and
- (4) $J_n \cap U(x_0)$ is a smooth curve.

This is possible since the singular points of J are isolated, and since $\lambda(x)$ is holomorphic away from singular points of J . The fact that $\lambda'(x) \neq 0$ in $U(x_0)$ means that λ is a local diffeomorphism in $U(x_0)$. Thus, shrinking $U(x_0)$ if necessary, we can insure that $J_n \cap U(x_0) = \lambda^{-1}(S^1) \cap U(x_0)$ has a single component. Let us write $J_n \cap U(x_0)$ as $\eta : [0, 1] \rightarrow U(x_0)$. Let

$$t_1 = \inf\{t \in [0, 1] \mid \eta(t) \in \Sigma\},$$

and

$$t_2 = \sup\{t \in [0, 1] \mid \eta(t) \in \Sigma\}.$$

Now pick x_1 and x_2 in $U(x_0)$ satisfying:

- (1) $x_i = \eta(s_i)$ for $i = 1, 2$ with $t_1 < s_1 < s_2 < t_2$, and
- (2) $\lambda(x_i)$ satisfies the Diophantine condition (3), for $i = 1, 2$.

This is possible since Diophantine points are dense in J_n .

Just as we chose the path Γ for the point x_0 above, we now choose paths Γ_1 and Γ_2 for the points x_1 and x_2 . We thus obtain two families of invariant manifolds, one each along Γ_1 and Γ_2 . Call these families of manifolds M_1 and M_2 respectively. Since the manifolds vary in a C^1 smooth fashion, we see that shrinking $U(x_0)$ suitably, M_1 and M_2 each disconnect $U(x_0)$. Additionally, by further shrinking $U(x_0)$ if needed, we can assume that M_1 and M_2 are disjoint. Thus the two families and the boundary of $U(x_0)$ bound a compact set, V .

We claim that the invariant curve γ constructed in Lemma 2.1 cannot intersect $M_1 \cap U(x_0)$ or $M_2 \cap U(x_0)$. To see this we proceed as follows. For ease of notation let M represent either M_1 or M_2 . Assume by way of contradiction that γ intersects $M \cap U(x_0)$ at the point p . There are three cases:

- (1) p is in a stable manifold in M (a manifold corresponding to a point in J_s),
- (2) p is in a centre manifold in M (a manifold corresponding to a point in J_n), or
- (3) p is in an unstable manifold in M (a manifold corresponding to a point in J_u).

In the first case, $F^n(p)$ converges to a single fixed point. We have assumed, however, that $\{F^n(p)\}$ has more than one limit point. In the second case, p is a limit point of $\{F^n(p)\}$. We have assumed, however, that Ω is nonrecurrent. In the third case the preimages, $F^{-n}(p)$, remain in $M \cap U(x_0)$. By shrinking $U(x_0)$ we can assume that γ_0 does not intersect $U(x_0)$. The point p however is the image $F^n(p')$ for some $p' \in \gamma_0$ and some $n > 1$. Thus p' must be in $U(x_0)$, but we have assumed that it is not.

We know that $\eta([s_1, s_2])$ is contained in V , and that V contains a point q_1 in Σ . We also know that $\eta(t_1) \in \Sigma$ and $\eta(t_2) \in \Sigma$ are in the complement of V . The invariant curve γ comes arbitrarily close to both of these points. Thus γ must leave and return to V infinitely many times. Let U' be a small neighbourhood of J_n . Then $V \setminus \overline{U'}$ is compact, and thus γ has accumulation points in $V \setminus \overline{U'}$. This is a contradiction, since all limit points of $\{F^n\}$ in $U(x_0)$ lie on J_n .

We notice that if the normal eigenvalue is constant along J , and satisfies the Diophantine condition (3), then precisely the arguments above show that there cannot be more than one rank 0 limit map.

Remark 2.6. *An example of an automorphism with one limit map is given in section 5.1.1. At present we do not know if there are any maps with more than one rank 0 limit.*

2.3. One Limit Map. We make several comments about the case where there is only one rank 0 limit map.

Assume there is only one rank 0 limit map, h . Just as in the case where there is more than one rank 0 limit, $h(\Omega)$ is a fixed point.

If q is an isolated fixed point then q can be attractive, repulsive, saddle, semi repulsive, semi attractive or neutral. However, [Ued86] Proposition 4.1 shows that the only fixed points to which other points converge uniformly are the attractive, semi attractive or neutral ones. We can eliminate the possibility that the fixed point is attractive, since in this case it would be in the interior of the Fatou component, and the Fatou component would thus be recurrent.

If q lies on a curve of fixed points we can locally write the equation for F in the form of Equation 1. Proposition 4.1 from [Ued86] and the fact that q is on the curve of fixed points show that q can only be semi attractive or neutral. The arguments in Section 2.2 show that q is a neutral fixed point.

3. THE RANK 1 CASE

We define F , Ω and Σ as in Section 2. In this section we allow elements of Σ to have rank 0 or 1. We focus our attention mainly on rank 1 elements of Σ .

3.1. Fibres of h . Let $h := \lim_{k \rightarrow \infty} (F|_{\Omega})^{n_k}$ be a generically rank 1 map. We make several elementary comments about the fibres of h .

DEFINITION 3.1 (V_q^h). Let $V_q^h := \{p \in \Omega \mid h(p) = q\}$. We may suppress the superscript h if it is clear from the context.

LEMMA 3.2. Fix $q \in h(\Omega)$. Let V'_q be the pure one dimensional irreducible components of V_q . Let $N_q := \{p \in V'_q \mid Dh(p) = 0\}$. Let V be an irreducible component of V'_q . If $s \in N_q \cap V$ then either

- (1) s is isolated in $N_q \cap V$, or
- (2) $V \subset N_q$.

Proof. Since V is an analytic variety and Dh is a holomorphic function on V , the zeros of Dh on V are either isolated or form an open set in V . In other words, s is isolated in $N_q \cap V$ or $V \subset N_q$. □

We also notice that at points of $p \in \Omega$ where the rank of Dh is 1, the image of h in a neighbourhood of p is smooth. This is regardless of whether p lies on a fibre of h which contains points where rank Dh is 0.

Finally we note that $F(V_q^h) = V_{F(q)}^h$, or in other words $h \circ F = F \circ h$. This implies that F is in some sense an automorphism of $h(\Omega)$.

3.2. Some Functional Relationships.

LEMMA 3.3. Let h_1 and h_2 be two rank 1 limit maps of $\{(F|_{\Omega})^n\}_{n=1}^{\infty}$:

$$h_1 = \lim_{k \rightarrow \infty} (F|_{\Omega})^{m_k},$$

and

$$h_2 = \lim_{k \rightarrow \infty} (F|_{\Omega})^{n_k}.$$

If $\{m_k - n_k\}$ is a finite subset of \mathbb{Z} , then h_1 and h_2 have the same fibres, and in fact $h_1 = F^l \circ h_2$ for some l .

Proof. Assume without loss of generality that $m_k > n_k$ and pick subsequences on $\{m_k\}$ and $\{n_k\}$ so that $m_k - n_k = l$. Then $F^{m_k} = F^{n_k} \circ F^l$. Taking limits we see that $h_1 = F^l \circ h_2$. □

Given $\{m_k\}$ and $\{n_k\}$, the above holds if we can find any subsequences of $\{m_{k_j}\}$ and $\{n_{k_j}\}$ whose differences, $\{m_{k_j} - n_{k_j}\}$, are a finite subset of \mathbb{Z} .

If we can find two subsequences of $\{m_k\}$ and $\{n_k\}$ such that (abusing notation by not including further subscripts)

$$m_k - n_k = l_1,$$

and

$$m_j - n_j = l_2,$$

and $l_1 \neq l_2$ then by the lemma we have

$$h_1 = F^{l_1} \circ h_2,$$

and

$$h_1 = F^{l_2} \circ h_2.$$

This in turn implies that $h_2 = F^{l_1 - l_2} h_2$, i.e. all points in $h_2(\Omega)$ are periodic.

3.3. Extending and Composing Limit Maps. In this section we describe a formal but natural method of extending h from Ω to $h(\Omega)$ continuously along orbits.

Write $\Omega = \cup_{n=1}^{\infty} \Omega_n$ where $\Omega_n \subset \subset \Omega_{n+1} \subset \subset \Omega$. Let h_1 and h_2 be two limit maps of $\{(F|_{\Omega})^n\}_{n=1}^{\infty}$:

$$h_1 = \lim_{k \rightarrow \infty} (F|_{\Omega})^{m_k}$$

and

$$h_2 = \lim_{k \rightarrow \infty} (F|_{\Omega})^{n_k}.$$

We define $h_2 \circ h_1$ as follows. Fix $\{n_k\}$. For each k choose $k' = k'(k)$ so that

$$\begin{aligned} |F^{n_k + m_{k'}} - F^{n_k} \circ h_1|_{\Omega_k} &= |F^{m_{k'}} \circ F^{n_k} - h_1 \circ F^{n_k}|_{\Omega_k} \\ &< \frac{1}{k}. \end{aligned}$$

Passing to a suitable subsequence define $h_2 \circ h_1 := \lim_{k \rightarrow \infty} (F|_{\Omega})^{n_{k_j} + m_{k'_j}}$.

LEMMA 3.4. *The map $h_2 \circ h_1$ is constant on fibres of h_1 .*

Proof. For ease of notation, we suppress the subscript j above:

$$h_2 \circ h_1 = \lim_{k \rightarrow \infty} (F|_{\Omega})^{n_k + m_{k'}}.$$

Let $h_1(p) = h_1(p')$. Then

$$\begin{aligned} |h_2 \circ h_1(p) - h_2 \circ h_1(p')| &\leq |h_2 \circ h_1(p) - F^{n_k + m_{k'}}(p)| + |F^{n_k + m_{k'}}(p) - F^{n_k + m_{k'}}(p')| \\ &\quad + |F^{n_k + m_{k'}}(p') - h_2 \circ h_1(p')|. \end{aligned}$$

We can make the first and third summands on the right hand side of the inequality as small as we like by choosing k large, since $(F|_{\Omega})^{n_k + m_{k'}} \rightarrow h_2 \circ h_1$ by definition. The second summand is less than the following.

$$|h_1 \circ F^{n_k}(p) - F^{n_k + m_{k'}}(p)| + |F^{n_k + m_{k'}}(p') - h_1 \circ F^{n_k}(p')| + |h_1 \circ F^{n_k}(p') - h_1 \circ F^{n_k}(p)|.$$

We have chosen Ω_k to be an exhaustion of Ω , so p and p' are in Ω_k for k large. Thus

$$(11) \quad \begin{aligned} |h_1 \circ F^{n_k}(p) - F^{n_k+m_{k'}}(p)| &= |F^{n_k+m_{k'}}(p) - F^{n_k} \circ h_1(p)| \\ &\leq |F^{n_k+m_{k'}} - F^{n_k} \circ h_1|_{\Omega_k} < \frac{1}{k}, \end{aligned}$$

and similarly for p' . We see that by choosing k large we can make the first and second summands as small as desired, and the third summand is 0 since $h_1(p) = h_1(p')$, and $F^{n_k} \circ h_1 = h_1 \circ F^{n_k}$. \square

LEMMA 3.5. *Our definition of $h_2 \circ h_1$ satisfies the following:*

$$h_2 \circ h_1(p) = \lim_{k \rightarrow \infty} F^{n_{k_j}} \circ h_1(p).$$

Proof. We have that $h_2 \circ h_1$

$$|h_2 \circ h_1(p) - F^{n_{k_j}} \circ h_1(p)| \leq |h_2 \circ h_1(p) - F^{n_{k_j}+m_{k'_j}}(p)| + |F^{n_{k_j}+m_{k'_j}}(p) - F^{n_{k_j}} \circ h_1(p)|.$$

By definition of $h_2 \circ h_1$ the first summand on the right hand side can be made as small as desired by making j large. Applying estimate (11) shows that the second summand can also be made small by choosing j large. \square

We make several notes about the above construction.

- (1) We do not know whether the maps $h_2 \circ h_1$ are unique: they might depend on the subsequence of $F^{n_k+m_{k'}}$ chosen.
- (2) For normality in the following setting we consider F^n as maps from $h_1(\Omega)$ to \mathbb{C}^2 , and we also allow limit maps to be infinite. Consider a small open set, U , in the immersed variety $h_1(\Omega)$. Its preimage, $h_1^{-1}(U) = V$, is an open set in Ω . Given a subsequence $\{F^{n_k}\}$, we pass to a convergent subsequence, giving us a limit map h_2 , possibly infinite. We define h_2 on U as the restriction of $h_2 \circ h_1$ to $V \cap \Omega_n$ for some n . Note that this definition is independent of the choice of n , by Lemma 3.4. Using this definition and Lemma 3.5 we see that $\{F^n\}$ is a normal family on $h(\Omega)$.
- (3) Given any rank 1 limit map h of $\{(F|_{\Omega})^n\}_{n=1}^{\infty}$, we can extend it to $h(\Omega)$ by considering $h \circ h$. We can then think of the limit map as being defined on the union of Ω and $h(\Omega)$.

3.4. Limit Varieties. In this section we show that the images of two distinct rank 1 limit maps cannot intersect except in one special case. We also show that the image of a limit map is locally irreducible. The images of limit maps are immersed varieties: Let $h(\Omega)$ be the image of the limit map h . For every point, $p \in h(\Omega)$, there is a neighbourhood, U , of p , and a connected component of $U \cap h(\Omega)$ containing p which is the zero set of a holomorphic function in U . For ease we shall call the images of the limit maps, limit varieties.

Proof of Theorem 1.2. To see that two limit varieties do not intersect in 0 dimensional sets we assume that they do. Call an intersection point p . In a small neighbourhood, W_p , of p

both $h(\Omega)$ and $g(\Omega)$ are one complex dimensional varieties, so we can say

$$h(\Omega) \cap W_p = \{\phi^{-1}(0)\},$$

for some holomorphic function $\phi : W_p \rightarrow \mathbb{C}$. (Here, since $g(\Omega)$ and $h(\Omega)$ are immersed varieties, we are only examining one component of $g(\Omega) \cap W_p$ and one component of $h(\Omega) \cap W_p$.) The restriction of ϕ to $g(\Omega) \cap W_p$ is either identically zero, or has isolated zeroes. In the former case $h(\Omega)$ and $g(\Omega)$ share an open set. In the latter case, ϕ restricted to any suitably small perturbation of $g(\Omega)$ also has zeroes. Thus if we perturb $g(\Omega)$ into Ω we see that $h(\Omega)$ contains points in Ω . This is a contradiction. \square

We see that by the same arguments a limit variety cannot come back and “hit” itself. It is also clear that it cannot locally self intersect: the image under h of a ball in Ω must be irreducible. Note that this does not imply that $h(\Omega)$ is a variety.

3.5. Fixed Points. We consider fixed points in the limit varieties of limit maps.

LEMMA 3.6. *Fixed points in the limit varieties of limit map cannot be attracting or repelling in \mathbb{C}^2 .*

Proof. Let $q \in h(\Omega)$ be a fixed point of F . For ease we let $q = (0,0)$. q cannot be an attractive fixed point; if it were it would be interior to the Fatou component.

Assume q is repelling. Let U be a small neighbourhood of q . Pick $p \in h^{-1}(\{q\}) \setminus U$, and recall that for j suitably large $F^{n_j}(p)$ is in U . Since under F^{-1} the origin is attractive, we see that the preimages of $F^{n_j}(p)$ remain in U ; i.e. p is not a preimage of $F^{n_j}(p)$. This contradiction shows that q is not repulsive. \square

If q lies on a curve of fixed points, then we see immediately from Section 3.1 of [Ued86] that we can also eliminate the possibility that q is a saddle. Additionally, as mentioned in Section 2.2, if the fixed point were semi attractive we would have normality of iterates of F in a neighbourhood of the fixed point. This is not possible since q is on the boundary of Ω .

4. POLYNOMIAL AUTOMORPHISMS

We would like to know which automorphisms of \mathbb{C}^2 have invariant nonrecurrent Fatou components. We show in this section that many polynomial automorphisms will not have such Fatou components.

The basis of our analysis is the paper by Friedland and Milnor, [FM86]. In this paper the authors show that any polynomial automorphism is conjugate to

- (1) an affine map,
- (2) a shear, $f(z, w) = (az + p(w), bw + c)$ with p a polynomial and $ab \neq 0$, or
- (3) a composition, $f_n \circ \dots \circ f_1$, of Hénon maps, $f_j(z, w) = (w, p_j(w) - a_j z)$ with p_j polynomials of degree at least two.

A simple examination of affine maps and shears shows that their Fatou components are either empty or all of \mathbb{C}^2 . If the Fatou component is all of \mathbb{C}^2 , then it is recurrent.

Hénon maps have finitely many fixed points ([FM86]) and thus cannot have two rank 0 limit maps.

By the work of Bedford and Smillie, [BS91a], if a Hénon map is hyperbolic, then the interior of the set of points with bounded forward orbits consists of a union of sinks. Thus these maps cannot have invariant nonrecurrent Fatou components with rank 1 limit maps.

We have proved Theorem 1.3.

5. EXAMPLES

We thank Berit Stensønes for providing invaluable help with these examples.

We note that Weickert [Wei98] as well as Buzzard and Forstneric [BF00] have also constructed automorphisms with prescribed jets. We not only ensure that the automorphisms are tangent to the identity at the origin and have the prescribed jet; we have constructed the automorphisms carefully to ensure that they leave the w -axis fixed as a set.

As mentioned in the Introduction, we construct three automorphisms: in Section 5.1.1 we construct an automorphism with one rank 0 limit, in Section 5.1.2 we construct an automorphism with one rank 1 limit, in Section 5.1.3 we construct an automorphism with multiple rank 1 limits. Their dynamics are examined in 5.2.1, 5.2.2 and 5.2.3 respectively.

5.1. Maps. We begin with four maps:

$$\begin{aligned} F_1(z, w) &= (z, w + z), \\ F_2(z, w) &= (ze^w, w), \\ F_3(z, w) &= (z, w - z), \text{ and} \\ F_4(z, w) &= (ze^{-w}, w). \end{aligned}$$

We also introduce

$$\begin{aligned} b_l(z, w) &= (z, z^l w), \text{ and} \\ b_l^{-1}(z, w) &= (z, z^{-l} w), \end{aligned}$$

with $l \in \mathbb{Z}^+$.

Note that b_l is holomorphic and one to one on $\mathbb{C}^* \times \mathbb{C}$. It also maps $\mathbb{C}^* \times \mathbb{C}$ onto itself.

We let $\pi_1(z, w) = z$ and $\pi_2(z, w) = w$.

We see that

$$G(z, w) := F_4 \circ F_3 \circ F_2 \circ F_1(z, w) = (ze^{ze^{z+w}}, w + z - ze^{z+w}).$$

We would like to remove all pure z terms up to order $l + 1$ from $\pi_2(G(z, w))$. To do so we construct a map $F_5(z, w) = (z, w + g(z))$ with $g(z) = a_2 z^2 + \dots + a_{l+1} z^{l+1}$, where we inductively choose the a_i to get rid of pure z terms of degree i . As an example (which we shall actually use later) we calculate this explicitly for $l = 2$.

We have that $\pi_2(G(z, w))$ is

$$w + z - ze^{z+w} = w - z^2 - \frac{z^3}{2!} - zw - z^2w - \frac{z^3w}{2!} - \frac{zw^2}{2!} + h_1(z, w),$$

with

$$h_1(z, w) = -z \left(\frac{w^3}{3!} + \frac{z^3}{3!} + \frac{zw^2}{2!} + \sum_{k=4}^{\infty} \frac{(z+w)^k}{k!} \right).$$

(Note that h_1 does not include all of the terms which are not pure z terms. We are doing this because we will need these not pure z terms later.)

We have that $(\pi_1(G(z, w)))^2$ is

$$z^2 e^{2ze^{z+w}} = z^2 + 2z^3 + 2z^3w + h_2(z, w),$$

with

$$h_2(z, w) = z^2 \left(2z^2 + 2z \sum_{k=2}^{\infty} \frac{(z+w)^k}{k!} + \sum_{k=2}^{\infty} \frac{(2z)^k e^{k(z+w)}}{k!} \right).$$

Similarly $(\pi_1(G(z, w)))^3$ is

$$z^3 e^{3ze^{z+w}} = z^3 + h_3(z, w),$$

with

$$h_3(z, w) = z^3 \left(\sum_{k=1}^{\infty} \frac{(3z)^k e^{k(z+w)}}{k!} \right).$$

When we add $(\pi_1(G(z, w)))^2$ to $\pi_2(G(z, w))$ we get

$$w + \frac{3z^3}{2} - zw - z^2w + \frac{3z^3w}{2} - \frac{zw^2}{2} + h_1(z, w) + h_2(z, w).$$

Now we subtract $3/2(\pi_1(G(z, w)))^3$ to $\pi_2(G(z, w))$ to get

$$w - zw - z^2w + \mathcal{O}(z^4, z^3w, zw^2).$$

For general l we see that $\pi_2(F_5 \circ G(z, w))$ becomes

$$w + \mathcal{O}(z^{l+2}, zw).$$

In Sections 5.1.1 through 5.1.3 we modify the above map. We shall see in Sections 5.2.1 through 5.2.3 that these new maps have the desired dynamical properties.

5.1.1. *Rank 0 Example.* We rewrite $\pi_2(F_5 \circ G(z, w))$ as

$$w + w \sum_{k=1}^{l+1} b_k z^k + \mathcal{O}(z^{l+2}, z^{l+2}w, zw^2).$$

We add a map $F_{5a}(z, w) = (z, we^{f(z)})$ where $f(z) = c_1 z + \dots + c_{l+1} z^{l+1}$. We can choose the c_k in such a way as to remove all terms of the form wz^j for $j = 1, \dots, l+1$ in $F_{5a} \circ G(z, w)$.

We add a further map, $F_6(z, w) = (z, we^{(l+1)z})$. Let

$$\tilde{H}(z, w) = F_6 \circ F_{5a} \circ F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1(z, w).$$

We have that

$$\tilde{H}(z, w) = (z + z^2 + \mathcal{O}(z^3, z^2w), (w + \mathcal{O}(z^{l+2}, z^{l+2}w, zw^2))e^{(l+1)ze^{ze^z+w}}).$$

Notice that $F_i(0, w) = (0, w)$ for $i = 1, \dots, 6$ so $\tilde{H}(0, w) = (0, w)$. Then $\tilde{H} \circ b_l(z, w) : \mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}^* \times \mathbb{C}$ is well defined, one to one and onto. Thus we also have that $b_l^{-1} \circ \tilde{H} \circ b_l : \mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}^* \times \mathbb{C}$ is well defined, one to one and onto.

For $z \neq 0$ we have

$$b_l^{-1} \circ \tilde{H} \circ b_l(z, w) = (ze^{ze^z+z^l w}, (w + \mathcal{O}(z^2, z^{l+2}w, z^l w^2))e^{(l+1)ze^{ze^z+z^l w}}e^{-lze^{ze^z+z^l w}}).$$

We notice that $b_l^{-1} \circ \tilde{H} \circ b_l(z, w) \rightarrow (0, w)$ as $(z, w) \rightarrow (0, w)$. Thus defining

$$H(z, w) = \begin{cases} b_l^{-1} \circ \tilde{H} \circ b_l(z, w) & z \neq 0 \\ (0, w) & z = 0 \end{cases}$$

we see that H is an automorphism of \mathbb{C}^2 .

We expand H :

$$(12) \quad H(z, w) = (z + z^2 + \mathcal{O}(z^3, z^{l+2}w, z^{2l+2}w^2), w + wz + \mathcal{O}(z^2, z^2w, z^l w^2)).$$

We note for future reference some of the computations leading to this:

$$\begin{aligned} ze^{ze^z+z^l w} &= z + z^2 + \mathcal{O}(z^3, z^{l+2}w, z^{2l+2}w^2). \\ ze^{ze^z+z^l w} &= z + z^2 + \mathcal{O}(z^3, z^{l+1}w, z^{2l+1}w^2). \end{aligned}$$

Putting these together we see

$$e^{(l+1)ze^{ze^z+z^l w}}e^{-lze^{ze^z+z^l w}} = 1 + z + \frac{3z^2}{2} + \mathcal{O}(z^3, z^{l+1}w, z^{2l+1}w^2).$$

It is now much easier to see where Equation (12) comes from. As we shall see in Section 5.1.2, by playing a little bit with the shears we can change the final form of H .

5.1.2. *Rank 1 Example.* To construct our example we use the maps F_1 through F_4 . We alter F_5 slightly, by making it remove all pure z^4 terms as well as z^2 and z^3 terms. (This was discussed in Section 5.1.) We leave out the map F_{5a} . We also leave out F_6 , for the moment. (We will return F_6 to exactly the same place it was before; we work backwards to see where the desired F_6 comes from.)

$$b_2^{-1} \circ F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1 \circ b_2(z, w) = (ze^{ze^z+z^2w}, (w - zw - z^2w + \mathcal{O}(z^3, z^3w, z^3w^2))e^{-2ze^z+z^2w}).$$

We notice the following:

$$(w - zw - z^2w)e^z = w - \frac{3z^2w}{2} + \mathcal{O}(z^3w).$$

We would like to duplicate this behaviour in our maps. To do so we add another overshear, F_6 .

$$F_6(z, w) = (z, we^{3z}).$$

Using calculations from Section 5.1.1 we see that

$$e^{-2ze^z+z^2w} e^{3ze^z+z^2w} = 1 + z + \frac{3z^2}{2} + \mathcal{O}(z^3, z^3w, z^5w^2).$$

This is not quite equal to e^z , but is close enough:

$$\begin{aligned} (w - zw - z^2w)(1 + z + \frac{3z^2}{2} + \mathcal{O}(z^3, z^3w, z^5w^2)) \\ = w - \frac{z^2w}{2} + \mathcal{O}(z^3w, z^3w^2). \end{aligned}$$

Thus we have

$$\begin{aligned} b_2^{-1} \circ F_6 \circ F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1 \circ b_2(z, w) \\ = (ze^{ze^z+z^2w}, w - \frac{z^2w}{2} + \mathcal{O}(z^3, z^3w, z^3w^2)). \end{aligned}$$

5.1.3. *Rotation Examples.* To construct this example, we use the automorphism constructed in Section 5.1.2 and add a rotation. We define H as

$$H(z, w) = \Theta_0 \circ b_2^{-1} \circ F_6 \circ F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1 \circ b_2(z, w)$$

where $\Theta_0(z, w) = (z, e^{i\theta_0}w)$. So we have

$$H(z, w) = (ze^{ze^z+z^2w}, e^{i\theta_0}(w - \frac{z^2w}{2} + \mathcal{O}(z^3, z^3w, z^3w^2))).$$

5.2. **Dynamics.** We note that the calculations in this section are similar to these carried out by Weickert [Wei98] and Hakim [Hak98].

5.2.1. *Rank 0 Example.* We show that H has a Fatou component, Ω , with the following properties:

- (1) $\Omega \neq \mathbb{C}^2$,
- (2) $\lim_{n \rightarrow \infty} H^n(p) = (0, 0)$ for all $p \in \Omega$, and
- (3) $(0, 0)$ is in the boundary of Ω .

For ease of computation we specifically choose $l = 2$. For convenience we use the following notation:

$$\begin{aligned} z_n &:= \pi_1(H^n(z, w)), \text{ and} \\ w_n &:= \pi_2(H^n(z, w)). \end{aligned}$$

We recall the expansion of H :

$$(z + z^2 + \mathcal{O}(z^3, z^4w, z^6w^2), w + wz + \mathcal{O}(z^2, z^2w, z^2w^2)).$$

We change coordinates: $z \rightarrow \frac{-1}{z}$. In the new coordinates H becomes:

$$\left(-\frac{1}{\frac{-1}{z} + \frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}, \frac{w}{z^4}, \frac{w^2}{z^6}\right)}, w - \frac{w}{z} + \mathcal{O}\left(\frac{1}{z^2}, \frac{w}{z^2}, \frac{w^2}{z^2}\right) \right).$$

We examine H on the following set:

$$U_{N, M} := \{(z, w) \in \mathbb{C}^2 \mid \operatorname{Re}(z) > N, |w| < M\},$$

where N and M are large numbers. M is chosen arbitrarily and is fixed, N is increased as needed in the following, though only finitely many times. (More correctly we should write $N(M)$.) We often write U instead of $U_{N, M}$.

We examine the z coordinate first.

$$z_1 = \left(z + 1 + \frac{1-b}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \right)$$

with b a constant, and choosing a suitably large N , $\operatorname{Re}(z) > N$. Thus by choosing N suitably large we see that

$$\operatorname{Re}(z_1) > \operatorname{Re}(z)$$

and

$$\frac{1}{2} \leq |z_1| \leq |z| + 2.$$

Assuming that U is invariant we have that

$$\frac{n}{2} \leq |z_n| \leq |z| + 2n.$$

Indeed, all that needs to be done to show that U is invariant is to show that $|w|$ remains less than M under iteration by H . We examine the w coordinate:

$$w_1 = w - \frac{w}{z} + \mathcal{O}\left(\frac{1}{z^2}, \frac{w}{z^2}, \frac{w^2}{z^2}\right).$$

We rewrite this slightly

$$w_1 = w \left(1 - \frac{1}{z} + \mathcal{O} \left(\frac{1}{z^2} \right) \right) + \mathcal{O} \left(\frac{1}{z^2} \right).$$

Again by choosing N large we can make $(1 - \frac{1}{z} + \mathcal{O}(\frac{1}{z^2}))$ less than 1, say

$$\left| \left(1 - \frac{1}{z} + \mathcal{O} \left(\frac{1}{z^2} \right) \right) \right| < 1 - \epsilon.$$

Then

$$\begin{aligned} |w_1| &= \left| w \left(1 - \frac{1}{z} + \mathcal{O} \left(\frac{1}{z^2} \right) \right) + \mathcal{O} \left(\frac{1}{z^2} \right) \right| \\ &\leq |w|(1 - \epsilon) + \left| \mathcal{O} \left(\frac{1}{z^2} \right) \right|. \end{aligned}$$

We note that $|\mathcal{O}(\frac{1}{z^2})|$ is bounded on $\text{Re}(z) > N$, and decreases to 0 as N increases to ∞ . Let

$$\alpha := \frac{|\mathcal{O}(1/z^2)|}{\epsilon}.$$

Choose N so large that the following hold:

- (1) $\alpha \ll M$, and
- (2) $\alpha + \epsilon M < M$.

Then if $|w| > \alpha$ we have that

$$|w|(1 - \epsilon) + \left| \mathcal{O} \left(\frac{1}{z^2} \right) \right| \leq |w|(1 - \epsilon) + |w|\epsilon \leq |w|.$$

If, on the other hand, $|w| < \alpha$ then

$$\begin{aligned} |w|(1 - \epsilon) + \left| \mathcal{O} \left(\frac{1}{z^2} \right) \right| &\leq |w| - \epsilon|w| + \alpha\epsilon \\ &< |w| + \epsilon M \\ &< \alpha + \epsilon M \\ &< M. \end{aligned}$$

So we indeed have that U is forward invariant under H .

We see that $z_n \rightarrow \infty$. Now all that remains to be shown is that $w_n \rightarrow 0$. To do so we examine the iterates of the w coordinate slightly more carefully.

$$\begin{aligned} w_1 &= w \left(1 - \frac{1}{z} + \mathcal{O} \left(\frac{1}{z^2} \right) \right) + \mathcal{O} \left(\frac{1}{z^2} \right), \\ w_2 &= w_1 \left(1 - \frac{1}{z_1} + \mathcal{O} \left(\frac{1}{z_1^2} \right) \right) + \mathcal{O} \left(\frac{1}{z_1^2} \right) \\ &= w \left(1 - \frac{1}{z} + \mathcal{O} \left(\frac{1}{z^2} \right) \right) \left(1 - \frac{1}{z_1} + \mathcal{O} \left(\frac{1}{z_1^2} \right) \right) \\ &\quad + \mathcal{O} \left(\frac{1}{z^2} \right) \left(1 - \frac{1}{z_1} + \mathcal{O} \left(\frac{1}{z_1^2} \right) \right) + \mathcal{O} \left(\frac{1}{z_1^2} \right). \end{aligned}$$

In general w_n is

$$w_n = w \prod_{j=0}^{n-1} \left(1 - \frac{1}{z_j} + \mathcal{O} \left(\frac{1}{z_j^2} \right) \right) + \sum_{j=0}^{n-1} \mathcal{O} \left(\frac{1}{z_j^2} \right) \prod_{k=j+1}^{n-1} \left(1 - \frac{1}{z_k} + \mathcal{O} \left(\frac{1}{z_k^2} \right) \right).$$

We examine the product

$$(13) \quad \prod_{j=0}^{n-1} \left(1 - \frac{1}{z_j} + \mathcal{O} \left(\frac{1}{z_j^2} \right) \right).$$

To understand convergence of this product we look at convergence of the series

$$(14) \quad \sum_{j=0}^{n-1} \log \left(1 - \frac{1}{z_j} + \mathcal{O} \left(\frac{1}{z_j^2} \right) \right).$$

But

$$\log \left(1 - \frac{1}{z_j} + \mathcal{O} \left(\frac{1}{z_j^2} \right) \right) = -\frac{1}{z_j} + \mathcal{O} \left(\frac{1}{z_j^2} \right).$$

Choosing N large we can insure that

$$\left| -\frac{1}{z_j} + \mathcal{O} \left(\frac{1}{z_j^2} \right) \right| > \frac{1}{2|z_j|},$$

and thus that the sum in (14) tends to negative infinity. Thus finally the product in (13) tends to 0.

We now turn our attention to the sum

$$\sum_{j=0}^{n-1} \mathcal{O} \left(\frac{1}{z_j^2} \right) \prod_{k=j+1}^{n-1} \left(1 - \frac{1}{z_k} + \mathcal{O} \left(\frac{1}{z_k^2} \right) \right)$$

which we rewrite as

$$(15) \quad \sum_{j=0}^{\infty} \alpha_j^n \mathcal{O} \left(\frac{1}{z_j^2} \right)$$

with

$$\alpha_j^n = \begin{cases} \prod_{k=j+1}^{n-1} \left(1 - \frac{1}{z_k} + \mathcal{O}\left(\frac{1}{z_k^2}\right)\right) & j \leq n-1 \\ 0 & j > n-1. \end{cases}$$

We recall that $|z_j| \geq j/2$, so

$$\left| \mathcal{O}\left(\frac{1}{z_j^2}\right) \right| \leq \frac{C}{(j/2)^2}.$$

Clearly then

$$\sum_{j=0}^{\infty} \mathcal{O}\left(\frac{1}{z_j^2}\right)$$

converges, and then since $|\alpha_j^n| < 1$ for all j and all n we have that (15) converges as well. Given $\epsilon > 0$ choose P so that

$$\sum_{j=P+1}^{\infty} \left| \mathcal{O}\left(\frac{1}{z_j^2}\right) \right| < \frac{\epsilon}{2}.$$

Choose n large enough that

$$\sum_{j=0}^P \left| \alpha_j^n \mathcal{O}\left(\frac{1}{z_j^2}\right) \right| < \frac{\epsilon}{2}.$$

Then we have that

$$\sum_{j=0}^{\infty} \left| \alpha_j^n \mathcal{O}\left(\frac{1}{z_j^2}\right) \right| < \epsilon.$$

We have proved that (15) is in fact 0. This and the fact that (13) converges to 0 show that $w_n \rightarrow 0$. Notice that the convergence of the infinite sums and products above is uniform on compacts in $U_{N,M}$.

We have thus succeeded in showing that $H^n(z, w) \rightarrow (\infty, 0)$, or in our original coordinates $H^n(z, w) \rightarrow (0, 0)$. This convergence is uniform on compacts, and holds at least on the open set

$$\cup_{M > > 1} \cup_{n=0}^{\infty} H^{-n}(U_{N(M), M}).$$

The above set is contained in a Fatou component, Ω . We notice several things about Ω .

- (1) Ω is not all of \mathbb{C}^2 . To see this recall that the w axis is fixed by H . Assume $\{H^n\}_{n=1}^{\infty}$ is normal in a neighbourhood of $(0, w)$ with w nonzero. In this case it is not possible for points arbitrarily close to $(0, w)$ to converge to $(0, 0)$. But this does in fact happen, precisely to points in $U_{N,M}$ which are close to $(0, w)$.
- (2) The full sequence H^n converges uniformly on compacts in Ω to $(0, 0)$. This statement is clearly true in $U_{N,M}$, and every point in Ω lands in some $U_{N,M}$ after sufficiently many iterations of H .

5.2.2. *Rank 1 Example.* We show that H has a Fatou component, Ω , with the following properties:

- (1) $\Omega \neq \mathbb{C}^2$,
- (2) $\lim_{n \rightarrow \infty} H^n$ is a rank generic 1 map, and
- (3) the w axis is in the boundary of Ω .

We use the same notation as in Section 5.2.1.

We recall the expansion of H :

$$(z + z^2 + \mathcal{O}(z^3, z^4w, z^6w^2), w - \frac{z^2w}{2} + \mathcal{O}(z^3, z^3w, z^3w^2)).$$

As before we change coordinates: $z \rightarrow \frac{1}{z}$. In the new coordinates H becomes:

$$\left(-\frac{1}{\frac{-1}{z} + \frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}, \frac{w}{z^4}, \frac{w^2}{z^6}\right)}, w - \frac{w}{2z^2} + \mathcal{O}\left(\frac{1}{z^3}, \frac{w}{z^3}, \frac{w^2}{z^3}\right) \right).$$

As in Section 5.2.1 we examine H on

$$U_{N,M} := \{(z, w) \in \mathbb{C}^2 \mid \operatorname{Re}(z) > N, |w| < M\}.$$

Exactly the same calculations as in Section 5.2.1 reveal that $z_n \rightarrow \infty$, again contingent on U being forward invariant.

We examine the w coordinate:

$$w_1 = w - \frac{w}{2z^2} + \mathcal{O}\left(\frac{1}{z^3}, \frac{w}{z^3}, \frac{w^2}{z^3}\right)$$

or

$$w_1 = w \left(1 - \frac{1}{2z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \right) + \mathcal{O}\left(\frac{1}{z^3}\right).$$

By repeating the arguments in Section 5.2.1 (with suitable modifications) we can see that $|w|$ remains bounded above by M under iteration by H . Thus U is indeed forward H invariant.

We look more closely at the iterates of the w coordinate. In general w_n is

$$w_n = w \prod_{j=0}^{n-1} \left(1 - \frac{1}{2z_j^2} + \mathcal{O}\left(\frac{1}{z_j^3}\right) \right) + \sum_{j=0}^{n-1} \mathcal{O}\left(\frac{1}{z_j^3}\right) \prod_{k=j+1}^{n-1} \left(1 - \frac{1}{2z_k^2} + \mathcal{O}\left(\frac{1}{z_k^3}\right) \right).$$

We examine

$$(16) \quad \prod_{j=0}^{n-1} \left(1 - \frac{1}{2z_j^2} + \mathcal{O}\left(\frac{1}{z_j^3}\right) \right).$$

To understand convergence of this product we examine

$$\sum_{j=0}^{n-1} \log \left(1 - \frac{1}{2z_j^2} + \mathcal{O}\left(\frac{1}{z_j^3}\right) \right) = \sum_{j=0}^{n-1} \left(\frac{-1}{2z_j^2} + \mathcal{O}\left(\frac{1}{z_j^3}\right) \right).$$

Choosing N large insures convergence of this series. We thus see that the product (16) converges to something finite.

We also see from this that the series

$$(17) \quad \sum_{j=0}^{n-1} \mathcal{O}\left(\frac{1}{z_j^3}\right) \prod_{k=j+1}^{n-1} \left(1 - \frac{1}{2z_k^2} + \mathcal{O}\left(\frac{1}{z_k^3}\right)\right)$$

converges and is in fact less than

$$\sum_{j=0}^{\infty} \mathcal{O}\left(\frac{1}{z_j^3}\right).$$

Again, convergence of all sums and products is uniform on compacts in $U_{N,M}$.

As in Section 5.2.1 we are examining a Fatou component, Ω , containing an open set

$$\cup_{M>>1} \cup_{n=0}^{\infty} H^{-n}(U_{N(M),M}).$$

We have again that convergence is uniform on compacts in Ω . We must show that the limit map is in fact rank 1. We note that having fixed an M and an $\epsilon > 0$, by choosing N suitably large we can insure that the quantities (16) and (17) each vary by less than ϵ in absolute value on $U_{N(M),M}$. Let h be the limit map $\lim_{n \rightarrow \infty} H^n$, and let (z, w) and (ζ, ω) be two points in $U_{N,M}$. For ease denote the product

$$\prod_{j=0}^{\infty} \left(1 - \frac{1}{2z_j^2} + \mathcal{O}\left(\frac{1}{z_j^3}\right)\right)$$

corresponding to (z, w) (resp. (ζ, ω)) by $P_{(z,w)}$ (resp. $P_{(\zeta,\omega)}$) and the limit

$$\lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} \mathcal{O}\left(\frac{1}{z_j^3}\right) \prod_{k=j+1}^{n-1} \left(1 - \frac{1}{2z_k^2} + \mathcal{O}\left(\frac{1}{z_k^3}\right)\right)$$

by $S_{(z,w)}$ (resp. $S_{(\zeta,\omega)}$). Then

$$\begin{aligned} |h(z, w) - h(\zeta, \omega)| &\geq |wP_{(z,w)} - \omega P_{(\zeta,\omega)}| - |S_{(z,w)} - S_{(\zeta,\omega)}| \\ &\geq |(w - \omega)P_{(z,w)}| - |\omega(P_{(z,w)} - P_{(\zeta,\omega)})| - |S_{(z,w)} - S_{(\zeta,\omega)}|. \end{aligned}$$

Letting $\omega = 0$, $z = \zeta$, and choosing w very large (perhaps enlarging M and N) we see that the last term is positive. Thus $h(z, 0)$ is not equal to $h(z, w)$. (Note that $P_{(z,w)}$ is bounded away from 0, so this is in fact possible.) Returning to the original coordinates we see that any point in Ω converges to a point on the w axis under iteration by H .

Next we show that $h(\Omega)$ is the entire w axis. Let $R > 0$ be a large real number. If we restrict $|w|$ to be less than $2R$, we can choose z_0 so small that (16) is very close to 1 in modulus and (17) is very small for all $|w| < 2R$. Then for $(z_0, w) \in \Omega$ with $|w| < 2R$ and z_0 suitably chosen, we have

$$|\pi_2(h(z_0, w)) - w| < 1.$$

Restricting our attention to $\{z_0\} \times \{|w| < 2R\}$ we think of h as a holomorphic function of one variable:

$$h : \{z_0\} \times \{|w| < 2R\} \rightarrow \{0\} \times \mathbb{C}.$$

Now an application of the argument principle shows that

$$\{0\} \times B(0, R) \subset h(\{z_0\} \times \{|w| < 2R\}).$$

Letting R increase we see that the w axis is contained in $h(\Omega)$.

Finally we show that the w axis is on the boundary of the Fatou component Ω . Assume $(0, w)$ is in Ω . Then we know that high enough iterates of a small neighbourhood of $(0, w)$ are very close to the w axis:

$$H^n(B(0, \delta) \times B(w, \delta)) \subset B(0, \epsilon) \times B(\pi_2(h(0, w)), \epsilon)$$

for all n larger than n' . If we change coordinates, $z \rightarrow -1/z$, we see that for $n > n'$ the z coordinate of H^n remains outside a large ball centered at the origin.

We now notice that the estimates we made about the growth of the z coordinate under iteration by H depended on the modulus of z being large, and on the w coordinate remaining bounded. In the present setting we have met all of these requirements. Thus we see that the real part of the z coordinate grows by roughly 1 for each iteration of H :

$$\operatorname{Re}(z_1) = \operatorname{Re}(z) + 1 + \mathcal{O}\left(\frac{1}{z}\right).$$

We have that the z coordinate remains in the complement of $B(0, 1/\epsilon)$, but we know that if this is the case then the real part of z grows by 1 under each iteration of H . Thus, after a finite number of iterates some point in the complement of $B(0, 1/\epsilon)$ moves into $B(0, 1/\epsilon)$. This is a contradiction.

5.2.3. Rotation Examples. The dynamics of this map is essentially the same as the dynamics of the rank 1 map. The difference, of course, is Θ_0 . We note first that the extra multiplicative factor $e^{i\theta_0}$ has no effect on the estimates in Sections 5.2.1 and 5.2.2. The rotation, however, does make the full sequence, $\{H^n\}$, not convergent.

If θ_0 is a rational multiple of 2π we obtain finitely many limit maps of H^n . Each such map has as its image the w axis. H acts as periodic rotation on the w axis, and the limit maps differ from each other by composition with H^j , for some j .

If θ_0 is an irrational multiple of 2π we obtain infinitely many limit maps of H^n . Each such map has as its image the w axis. H acts as an irrational rotation on the w axis, and the limit maps differ from each other by composition with H^j , for some j .

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